Math 53: Multivariable Calculus **Sections 102, 103**

Worksheet for 2020-03-30

Computations

Problem 1. Firstly, a purely computational problem with no tricks whatsoever: evaluate the following triple integral.

$$\int_1^2 \int_0^{2z} \int_0^{\ln x} x e^{-y} \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z.$$

Problem 2. Consider the region above the cone $z = \sqrt{3x^2 + 3y^2}$ and below the sphere $z = \sqrt{9 - x^2 - y^2}$. Set up an integral for the volume of this region using each of the following integration orders:

(a) $\mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y$
(b) $\mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$
(c) $\mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta$
(d) $\mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}\rho$
(e) $\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$ (you will need two integrals)

Problem 3. A stick of length 1 (and no thickness) is randomly dropped on a ruled sheet of paper, where the lines are spaced 2 units apart. What's the probability that the stick touches a line?

(Let Y be a random variable taking values between -1 and 1, uniformly distributed. Let W be a random variable taking values between 0 and 2π, uniformly distributed. We can think of the stick as having one end with y-coordinate Y and the other end at an angle of W relative to the first end. In terms of Y and W, when does the stick cross the x-axis?)

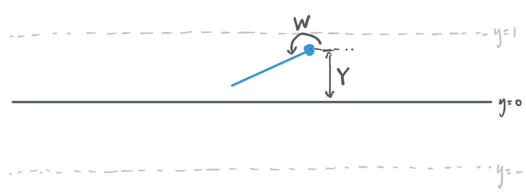

FIGURE 1. Problem 3. If Y is equally likely to be anywhere between -1 and 1, and the angle W is equally likely to be anywhere between 0 and 2π, what's the probability that the line segment crosses the line $y = 0$?

Problem 4. Alice and Bob are hoping to meet up sometime between 8AM and 9AM but did not decide on an exact time. Alice tends to be early; the function

$$f_A(x) = \begin{cases} 2 - 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

describes the probability density function for her time of arrival, where x is in "hours after 8AM."

On the other hand, Bob is not likely to wake up before 8AM, so the function

$$f_B(y) = \begin{cases} 2y & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

describes the probability density function for his time of arrival, where x is in "hours after 8AM."

Each of them will arrive, wait for 10 minutes, and then leave if the other person does not show up. What is the probability that they will meet successfully?

For some of the problems, I have provided answers so that you can check your work. For others, I have provided a sketch of how to solve the problem, in the hope that it is more useful than an answer, while still leaving you with the task of actually carrying it out. As always, I am willing to elaborate further on request.

Problem 1. 5/3

Problem 2.

(a) $\displaystyle\int_{-3/2}^{3/2}\int_{\sqrt{(9-4y^2)/4}}^{\sqrt{(9-4y^2)/4}}\int_{\sqrt{3x^2+3y^2}}^{\sqrt{9-x^2-y^2}} 1\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$

(b) $\displaystyle\int_0^{2\pi}\int_0^{3/2}\int_{r\sqrt{3}}^{\sqrt{9-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$

(c) $\displaystyle\int_0^{2\pi}\int_0^{\pi/6}\int_0^{3}\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$

(d) The bounds in the preceding part are all constants, so for this you can just interchange them without any worry.

(e) $\displaystyle\int_0^{3\sqrt{3}/2}\int_{-z/\sqrt{3}}^{z/\sqrt{3}}\int_{-\sqrt{(z^2-3y^2)/3}}^{\sqrt{(z^2-3y^2)/3}} 1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{3\sqrt{3}/2}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}}\int_{-\sqrt{9-y^2-z^2}}^{\sqrt{9-y^2-z^2}} 1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

Problem 3. The shaded region in the below diagram corresponds to situations where the stick touches the line. The probability distributions for Y and W are uniform, so the desired probability is just the area of the shaded region divided by the area of the whole rectangle.

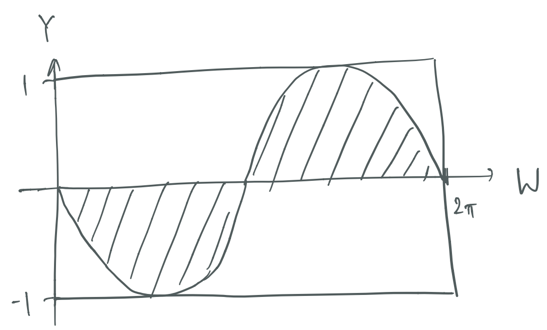

FIGURE 2. The curve is $Y + \sin W = 0$.

Problem 4. The shaded region in the below diagram corresponds to situations where Alice and Bob successfully meet. The joint probability distribution is $f(x, y) = (2 - 2x)(2y)$ on the square, so you need to integrate that over the shaded region to get the final answer. (It may be easiest to integrate it over the two non-shaded triangles instead, and then to subtract that from 1.)

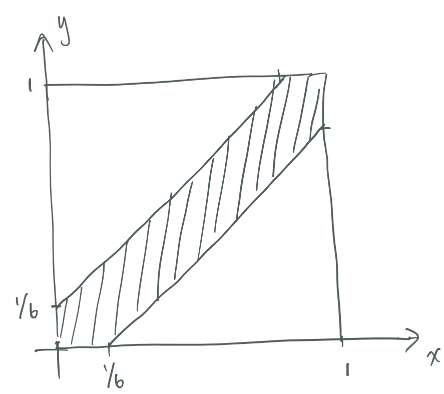

FIGURE 3. The lower right triangle represents if Alice shows up more than 10 minutes after Bob. The upper right triangle represents if Alice shows up more than 10 minutes before Bob. The middle shaded strip represents when they actually successfully meet up.